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GENERALIZED EQUATIONS OF ONE-DIMENSIONAL FILTRATION WITH FRACTIONAL-POWER DIFFERENTIATIONS

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Based on the formalism of fractional-power integrodifferentiation, the theory of filtration in porous media with fractal structure is generalized. Consideration is given to the cases of steady and unsteady filtration. For the steady filtration, nonlinear solutions are shown to exist. In the unsteady case, a new class of periodic solutions may exist.

1. Despite the long history of keen interest in investigations of liquid motion in porous media, in the filtration theory a number of problems still remain to be solved. The features of the spatial structure and the multiphase condition of the system lead to a complicated nature of the phenomena in them, and at present no adequate mathematical tool of quantitative consideration of the filtration flows in porous structures exists.

As has become apparent, for consideration of the geometry of pores it is necessary to use the spaces of fractal dimensions [1] and to represent a pore as a fractal. The profound physical nature that underlies the fractal state of a system consists of the fact that the characteristic spatial scale of inhomogeneity of the system can be compared to the characteristic microscopic spatial parameters, which, in turn, leads to the necessity of analyzing the properties within the framework of the concept of spatial nonlocalization [2]. The next feature, i.e., the multiphase state, also necessitates the development of a radically new approach to consideration of the properties of porous media. As is noted in [3], the share of the phase boundary in porous media is such that it makes a contribution to the quantities observed. The region between two phases represents a substance intermediate between the interacting phases in state; it is characterized by the presence of fluctuations, the complicated correlations due to spatial inhomogeneity, and the absence of the traditional structure of hierarchy of relaxation times. The complicated nature of the space and time correlations gives rise to memory and self-organization effects. By virtue of the reasons considered, the methods of quantum statistical physics of disordered media, such as the methods of *T*-matrix [3], percolations, and the approximation of a coherent potential [4] that are widely used in investigations of the properties of porous media, turn out to be inadequate. A description of the properties of systems with fractal structure is beyond the scope of the traditional methods of statistical physics.

Recent investigations of the properties of systems with fractal structure have formed a new concept in natural science, i.e., the concept of fractals [5, 6], of particular interest in which is the comparatively recent method based on the use of the formalism of fractional integrodifferentiation [7–12]. Despite the detailed development of the mathematical apparatus of fractional derivatives [13], their use in natural science was blocked by the absence of physical interpretation. It is shown in [7] that the use of a fractional derivative with respect to time allows one to take into account the memory effects which for a certain physical quantity

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$F(t)$ related to another physical quantity $f(t)$ indicates the presence of the relationship $F(t) = \int_0^t K(t-\tau)f(\tau)d\tau$ between them. For the Markovian process with a total absence of memory, the function $K(t)$ has the form $K(t-\tau) = \eta\delta(t-\tau)$, where η is a positive constant. In the case of total memory, $K(t) = t^{-1}\theta(t-\tau)$. When the time intervals with the memory effect form a set of the Hausdorff–Bezikovich measure (the Cantor set), the correlation between $F(t)$ and $f(t)$ is given by the fractional integral $F(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau$, where the exponent of the fractional integral α coincides with the fractal dimension of the Cantor set. Evolution of a system with partial memory is described by the fractional derivatives $\frac{\partial}{\partial t}f(t) \Rightarrow \frac{\partial^\alpha}{t_0\partial\tau^\alpha}f(\tau) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{t_0 d\tau}\int_0^\tau \frac{f(\eta)}{(\xi-\eta)^\alpha}d\eta$. As is seen in the definition of a fractional derivative, the sought function is under the time integral, which corresponds to accounting for the memory effects; the measure of participation of the values of the function at later times is determined by α, which precisely is the fractal dimension of the system. In the case of space variables, passage to a fractional derivative corresponds to accounting for space correlations, and it has a form similar to the fractional derivative with respect to time with the substitution $t_0 \to l_0$, $\tau \to \xi$.

The present work is devoted to an investigation of the filtration process based on the generalization of the equations with fractional-power differentiation that are suggested in [9].

2. The initial equations of filtration in porous media are of the form [14–16]

$$\frac{\partial\,(m\rho)}{\partial t} = \operatorname{div}\,(\rho\vec{v})\,,\quad \vec{v} = -\frac{k}{\mu}\operatorname{grad} P\,. \tag{1}$$

The system of equations (1) must be supplemented with the equation of state $\rho = \rho(P, T)$ and the equation of porosity $m = m(P, T) = m(\rho)$. It should be noted that, despite the wide use of the system of equations (1) and their various modifications [14–16], they involve difficulties if it is necessary to take into account the memory effects and space correlations and to explain the nonlinear properties of the system as a whole.

We generalize the system of equations (1) by using the definitions of a fractional derivative. Thereafter we will consider the one-dimensional problem

$$\frac{d^\alpha\,(m\rho)}{t_0 d\tau^\alpha} + \frac{1}{l_0}\frac{d^\beta}{d\xi^\beta}(\rho\nu\,(\xi)) = 0\,,\quad \nu = -\frac{k}{l_0\mu}\frac{d^\gamma}{d\xi^\gamma}P\,(\xi,\tau)\,. \tag{2}$$

Successive derivation of equations with fractional-power differentiation for different problems of heat and mass transfer is given in [7–12]. In the present work, the emphasis is on an analysis of the solutions of equations with fractional-power differentiation.

Now we dwell on consideration of steady filtration.

3. In the steady case, the system of equations (2) acquires the form

$$\frac{d^\beta}{d\xi^\beta}(\rho\nu\,(\xi)) = 0\,,\quad \nu = -\frac{k}{l\mu}\frac{d^\gamma}{d\xi^\gamma}P\,(\xi)\,. \tag{3}$$

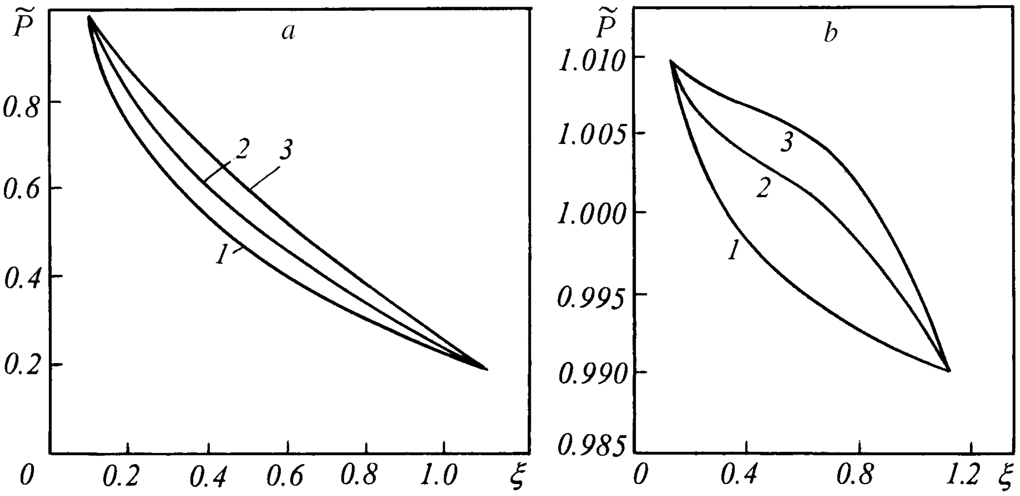

Fig. 1. Dimensionless pressure $\tilde{P} = P(\xi)/P_1$ vs. dimensionless coordinate ξ for $\alpha = 1+\gamma(a)$ and $\alpha = 2+\gamma(b)$: 1) $\gamma = 0.7$; 2) 0.8; 3) 0.9.

From (3) the following equation for pressure originates:

$$\frac{d^\beta}{d\xi^\beta}\left(\rho\,\frac{k}{l\mu}\,\frac{d^\gamma}{d\xi^\gamma}P\,(\xi)\right) = 0\,. \tag{4}$$

The solution of (4) essentially depends on the equation of state $\rho = \rho(P, T)$. In the case of a liquid that is considered in what follows we can set ρ = const. Then we have

$$\frac{d^{\beta+\gamma}}{d\xi^{\beta+\gamma}}P\,(\xi) = 0\,. \tag{5}$$

Introducing the notation $\alpha = \beta+\gamma$, we consider the case $1<\alpha\le 2$, $\alpha = 1+\eta$, $0<\eta\le 1$. The solution of (5) that satisfies the boundary conditions $P(\xi_1) = P_1$, $P(\xi_2) = P_2$, $\xi_1 = a/l$, $\xi_2 = b/l$, $l = b-a$ has the form

$$P\,(\xi) = (\xi_1\xi_2)^{1-\eta}\left\{(\xi_2^\eta P_1 - \xi_1^\eta P_2)\,\xi^{\eta-1} + (\xi_1^{\eta-1}P_2 - \xi_2^{\eta-1}P_1)\,\xi^\eta\right\}. \tag{6}$$

For $\eta = 1$, the solution (6) becomes the known solution of a one-dimensional problem. In Fig. 1a the dependence of the pressure on the dimensionless coordinate is shown for different α. Thus, for $0<\eta<1$ the solution is nonlinear. The radical difference of the fractal concept from the traditional approach lies in the possibility of considering Eq. (5) for the case $2<\alpha\le 3$. Then the solution of (5) acquires the form ($\alpha = 2+\gamma$, $0<\gamma\le 1$)

$$P\,(\xi) = \frac{a}{\Gamma\,(\gamma)}\,\xi^{\gamma-1} + \frac{b}{\Gamma\,(1+\gamma)}\,\xi^\gamma + \frac{d}{\Gamma\,(2+\gamma)}\,\xi^{1+\gamma}\,. \tag{7}$$

As a boundary condition, in addition to $P(\xi_1) = P_1$, $P(\xi_2) = P_2$, it is necessary to use one more condition. We consider the boundary condition $\frac{d^{1+\gamma}}{d\xi^{1+\gamma}}P(\xi)|_{\xi=\xi_1} = -\frac{\nu(\xi_1)L}{k}$, where $\nu(\xi_1)$ is the filtration rate at the boundary. The solution (7) acquires the form

$$P(\xi) = (\xi_1\xi_2)^{(1-\gamma)}\left\{\left(\tilde{P}_1\xi_2^{\gamma} - \tilde{P}_2\xi_1^{\gamma}\right)\xi^{\gamma-1} + \left(\tilde{P}_2\xi^{\gamma-1} - \tilde{P}_1\xi_2^{\gamma-1}\right)\xi^{\gamma}\right\} - \frac{\nu(\xi_1)L}{k\Gamma(1+\gamma)}\xi^{1+\gamma}, \tag{8}$$

where

$$\tilde{P}_1 = P_1 - \frac{\nu(\xi_1)L}{k\Gamma(2+\gamma)}\xi_1^{1+\gamma}; \quad \tilde{P}_2 = P_2 - \frac{\nu(\xi_1)L}{k\Gamma(2+\gamma)}\xi_2^{1+\gamma}$$

Figure 1b provides dependence (8) for different γ. Thus, within the framework of the concept of fractals, the nonlinear properties of filtration in porous media are explained from a radically different point of view. An investigation of the variant $\rho \neq \text{const}$ does not encounter major difficulties.

4. We consider the case of unsteady filtration. Simultaneous solution of Eqs. (2) leads to the following relation:

$$\frac{d^{\alpha}(m\rho)}{t_0 d\tau^{\alpha}} - \frac{1}{l_0}\frac{d^{\beta}}{d\xi^{\beta}}\left\{\frac{k}{\mu}\rho\frac{d^{\gamma}}{d\xi^{\gamma}}P(\xi,\tau)\right\} = 0. \tag{9}$$

Expression (9), together with the equation of state $\rho = \rho(P, T)$, represents a closed system of equations. In the case of an incompressible heavy fluid in a constant porous medium [14], the equation of motion (9) acquires the form

$$\frac{d^{\alpha}P(\xi,\tau)}{d\tau^{\alpha}} - \frac{bkt_0}{ml_0^2\mu}\frac{d^{\beta}}{d\xi^{\beta}}\left\{\frac{d^{\gamma}}{d\xi^{\gamma}}P(\xi,\tau)\right\} = 0. \tag{10}$$

Here b is the bulk modulus of the fluid. We consider the particular case $\beta = \gamma = 1$ when (10) can be written as

$$\frac{d^{\alpha}P(\xi,\tau)}{d\tau^{\alpha}} - A\frac{d^2}{d\xi^2}P(\xi,\tau) = 0, \quad A = \frac{bkt_0}{ml_0\mu}. \tag{11}$$

A general solution of Eq. (11) that satisfies the initial condition $P(\tau = 0, \xi) = P_0$ for an infinite medium is as follows:

$$P(\xi,\tau) = P_0\sqrt{\frac{\xi}{2\pi}}\,\tau^{\alpha-1}\int_0^{\infty} dk\,\sqrt{k}\,J_{-1/2}(k\xi)\,E_{\alpha,\alpha}(-A\tau^{\alpha}k^2), \tag{12}$$

where

$$E_{\alpha,\beta}(\xi) = \sum_{n=0}^{\infty}\frac{\xi^n}{\Gamma(\alpha n+\beta)}, \quad \alpha > 0, \quad \beta > 0.$$

If we set $\alpha = 1$ in (12), we arrive at the known solution $P(\xi, \tau) = \frac{P_0}{(4\pi At)^{1/2}}\exp(-\frac{\xi^2}{4A\tau})$. For the cases $\alpha = 1/2$ and $1/4$ in (12), instead of $E_{\alpha,\alpha}(-z^{\alpha})$, it is necessary to substitute the functions

$$E_{1/2,1/2}(-\sqrt{z}) = \frac{1}{\sqrt{\pi}} - \sqrt{z}\exp(z)\,[1 - \operatorname{erf}(\sqrt{z})]\,, \tag{13}$$

$$E_{1/4,1/4}(-z^{1/4}) = \frac{1}{\Gamma(1/4)}\,{}_1F_1(1;\,1/4;\,z) + \frac{\sqrt{z}}{\Gamma(3/4)}\,{}_1F_1(1;\,3/4;\,z) - z^{3/4}\exp(z)\,[1 + \operatorname{erf}(\sqrt{z})] - \frac{z^{1/4}}{\sqrt{\pi}}\,. \tag{14}$$

Having omitted a detailed analysis, we would like to note that at $0 < \alpha < 1$ the solution (12) is of periodic nature, which leads to wavy processes of filtration in porous media with steady initial conditions, i.e., to a new class of fluid flows in porous media.

5. Thus, within the framework of the suggested formalism of fractional integrodifferentiation, we are able to obtain the existing solutions and to carry out their nontrivial generalization. Passage to fractional derivatives actually means a radically new method of the quantitative description of processes which are not local with respect to time and space. Here, the fractal dimension becomes the most important quantity in the fractal concept. The fundamental property of the latter is that it determines the order of a fractional derivative and a fractional integral and is "the governing" parameter of a theory similar to this parameter in the physics of open systems [17]. It is pertinent to note that the fractal dimension as such can vary (multifractal condition). Of importance is the radical difference between the two "fitting" parameters t_0 and α. If with change in t_0 the functional form of the solution does not change, then with change in α, as is seen from relations (13) and (14), the functional form of the solution (12) changes. This allows us to explain the experimentally observed dependences with a high accuracy [11]. Here, the regularities of a change in the fractal dimension can serve as a basis for qualification of fractal systems.

NOTATION

$K(t)$, memory function; $\delta(t)$, Dirac delta function; $\theta(t)$, Heaviside unit function; $\Gamma(\zeta)$, Euler gamma function; t_0, characteristic time of the process under consideration; $\tau = t/t_0$, dimensionless time; l_0, characteristic space scale; $\xi = x/l_0$, dimensionless coordinate; k, penetration factor characterizing the filtration medium; μ, absolute coefficient of viscosity; v, filtration rate; $P(\xi)$, pressure; T, temperature; m, porosity; ρ, density; α, β, γ, fractal dimensions; $E_{\alpha,\beta}(z)$, Mittag–Leffler function; ${}_1F_1(a;\,b;\,z)$, degenerate hypergeometric Kummer function.

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